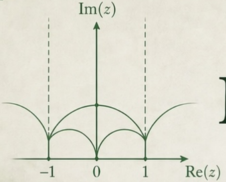


$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n$$

$$q = e^{2\pi i \tau},\ \mathrm{Im}(\tau) > 0$$

Introduction to

Modular Forms and Hecke Operators

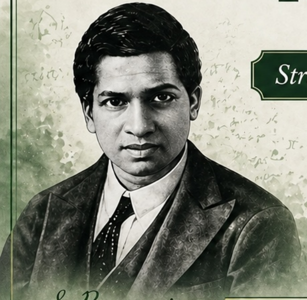
Structure and Arithmetic of Modular Forms

Srijan Raghunath

Dimitrios Nikolakopoulos (DRP Mentor)

Directed Reading Program (UConn)

May 2026



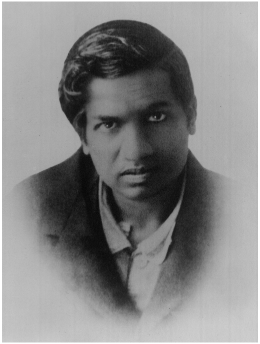
S. Ramanujan

— 1887–1920 —

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

E. Hecke

— 1887–1947 —



Srinivasa Ramanujan

Ramanujan defined

$$q\Big((1-q^1)(1-q^2)(1-q^3)(1-q^4)\dots\Big)^{24} := \sum_{n=1}^{\infty} \tau(n) q^n$$

First terms

$$q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \cdots$$

Values

n	1	2	3	4	5	6	7
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744

The Delta–function (or discriminant–function) defined as

$$\Delta(z) := q\Big((1-q^1)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\dots\Big)^{24}$$

or equivalently

$$\Delta(z) := q\prod_{m=1}^{\infty}(1-q^m)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n \ , \qquad q = e^{2\pi i z}$$

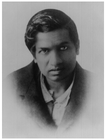
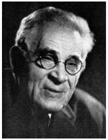
and is a **modular (cusp) form of weight 12 for $\mathbf{SL}_2(\mathbb{Z})$**.
For $\mathrm{Im}(z) > 0$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, this means that

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z).$$

Why are modular forms interesting?

- Arithmetic Geometry: Elliptic curves, BSD Conjecture,...
- Number Theory: Partitions, Quad. forms, ...
- Mathematical Physics: Mirror symmetry,...
- Representation Theory: Moonshine, symmetric groups,...

Historical Line: τ-Function and Modular Forms

1916 — Ramanujan

Introduces $\tau(n)$ and conjectures:

$$\boxed{\tau(mn) = \tau(m)\tau(n)}$$

$\gcd(m, n) = 1$

$$\boxed{\tau(p^\ell) = \tau(p)\tau(p^{\ell-1}) - p^{11}\tau(p^{\ell-2})}$$

1917 — Mordell

Proved multiplicativity and recursion

1937 — Hecke

Developed operator theory:

modular forms $\updownarrow$ eigenforms

1970s — Atkin–Lehner

newforms $\updownarrow$ modular forms into eigen–bases

Modular forms for $\mathrm{SL}_2(\mathbb{Z})$

Definition
A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ if:

- **Modularity condition**

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

- **Holomorphic at** ∞ ($\lim\limits_{z \to i\infty} f(z)$ exists)

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \qquad q = e^{2\pi i z}$$

$$M_k := M_k(\mathrm{SL}_2(\mathbb{Z})) = \left\{ \begin{matrix} \text{modular forms} \\ \text{of weight } k \end{matrix} \right\}$$

Fact: M_k forms a $\mathbb{C}-$vector space!

$f \in M_k$ & $g \in M_\ell \implies fg \in M_{k+\ell}$.

$\mathrm{SL}_2(\mathbb{Z})$ generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, it suffices that

$$f\left(-\frac{1}{z}\right) = z^k f(z) \quad \text{and} \quad f(z+1) = f(z).$$

For even $k \geqslant 4$:

$$G_k(z) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz+n)^k}$$

Eisenstein Series E_k

$$E_k(z) := \frac{G_k(z)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

where $\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$ and

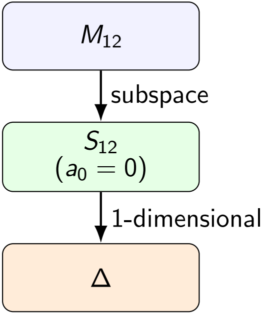
$$B_k = k^{\text{th}} - \text{Bernoulli number}.$$

Cusp Forms for $\mathrm{SL}_2(\mathbb{Z})$

Definition

$$S_k := S_k(\mathrm{SL}_2(\mathbb{Z})) = \{f \in M_k : a_0 = 0\}$$

the subspace of *cusp forms*.
(vanishes at infinity)



Key Example (weight 12)

$$\frac{E_4^3 - E_6^2}{1728}$$

is a cusp form of weight 12.

Key Result

All cusp forms of weight 12 are multiples of Δ.

$$\boxed{\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}}$$

Valence Formula

Valence formula

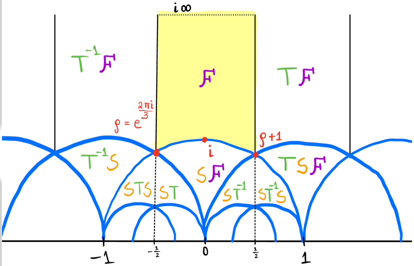
Let $f \not\equiv 0$ be a modular form of weight k. Then

$$v_\infty(f) + \frac{v_i(f)}{2} + \frac{v_\rho(f)}{3} + \sum_{p \in \mathcal{F} \setminus \{i, \rho\}} v_p(f) = \frac{k}{12}.$$

Fundamental domain $\mathcal{F}$

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leqslant \Re(z) \leqslant \frac{1}{2},\ |z| \geqslant 1 \right\}$$

- Every $z \in \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $w \in \mathcal{F}$.
- Interior points are inequivalent.



Dimension Formula

Basic facts

- If $k < 0$, then $M_k = \{0\}$
-

k	0	2	4	6	8	10
$\dim M_k$	1	0	1	1	1	1

Key isomorphism

$$M_{k-12} \xrightarrow{\cong} S_k, \quad g \mapsto g\,\Delta$$
$$\Longrightarrow \quad \dim S_k = \dim M_{k-12}$$

Decomposition

$$M_k = \mathbb{C}E_k \oplus S_k$$

$$f = a_f(0)\,E_k + \big(f - a_f(0)\,E_k\big)$$
$$\mathbb{C}E_k \cap S_k = \{0\},$$

so $\dim M_k = 1 + \dim S_k$.

Dimension recursion

$$\boxed{\dim M_k = 1 + \dim M_{k-12}}$$

Example

- $\dim S_{12} = \dim M_0 = 1$
- If $f \in S_{12}$, then $f = c\,\Delta$

Action of $\mathrm{SL}_2(\mathbb{Z})$ on X_m

The set X_m

$$X_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) : ad - bc = m \right\}$$

Group action

$$\mathrm{SL}_2(\mathbb{Z}) \curvearrowright X_m \quad \text{(left multiplication)}$$

Orbit representatives

Every orbit has a representative of the form:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = m, \quad 0 \leqslant b < d$$

$$X_m = \bigsqcup_{ad=m} \bigsqcup_{b=0}^{d-1} SL_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Slash operator

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $f \in M_k$:

$$(f|_k\gamma)(z) = (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

Properties

- Linearity: $(\lambda f)|_\gamma = \lambda(f|_\gamma)$
- Additivity: $(f+g)\big|_\gamma = f|_\gamma + g|_\gamma$
- Compatibility: $f|_{\gamma_1\gamma_2} = (f|_{\gamma 1})|_{\gamma 2}$
- Invariance: If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $f|_\gamma = f$

Hecke Operators

Recall the orbit representatives

$$SL_2(\mathbb{Z})\backslash X_m \;\cong\; \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m,\ 0 \leqslant b < d \right\}$$

Definition (Hecke operators)

For $f \in M_k$:

$$T_m[f] = m^{\frac{k}{2}-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b=0}^{d-1} f\Big|_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}$$

equivalently,

$$T_m[f](z) = m^{k-1} \sum_{ad=m} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{az+b}{d}\right).$$

Key fact:

$$T_m : M_k \longrightarrow M_k, \qquad T_m : S_k \longrightarrow S_k$$

Fourier Coefficients of Hecke Eigenforms

Let $f \in S_k$ be a Hecke eigenform
$T_m[f] = \lambda_m f$

Hecke action on coefficients
$$T_m[f](z) = \sum_{n=0}^{\infty} c(n)\, q^n$$
$$c(n) = \sum_{d\,|\,(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right)$$

Normalization
$a_f(1) \neq 0$
$$f(z) = q + \sum_{n \geqslant 2} a_f(n) q^n$$

Hecke action on coefficients
$$\sum_{d\,|\,\gcd(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right) = \lambda_m\, a_f(n)$$

Eigenvalues
$a_f(m) = \lambda_m$

Multiplicativity
$a_f(mn) = a_f(m)\, a_f(n)$
if $\gcd(m,n) = 1$

Hecke recurrence
$a_f(p^{\ell+1}) + p^{k-1}\, a_f(p^{\ell-1}) = a_f(p)\, a_f(p^{\ell})$

Properties of Hecke Operators

- If $\gcd(m,n)=1$, then $T_m \circ T_n = T_{mn}$.
  Sketch of proof: Comparing the coefficients $T_m\big(T_n[f]\big) = T_{mn}[f]$.

- Let p be prime and $\ell \geqslant 1$, $T_{p^\ell} \circ T_p = T_{p^{\ell+1}} + p^{k-1} T_{p^{\ell-1}}$.
  Sketch of proof: $\widetilde{T_{p^\ell}} := \sum_{j=0}^{\ell} \sum_{b=0}^{p^j-1} \begin{pmatrix} p^{\ell-j} & b \\ 0 & p^j \end{pmatrix}$, so
  $p^{\ell(1-\frac{k}{2})}\ T_{p^\ell}[f] = f\Big|_{\widetilde{T_{p^\ell}}}$.

- For all $m,n \in \mathbb{N}$, $T_m \circ T_n = \sum_{d \mid \gcd(m,n)} d^{k-1} T_{\frac{mn}{d^2}}$.
  Sketch of proof: Enough to prove it for prime powers.

Hecke Eigenform Properties

Let $f = q + \sum_{n \geqslant 1} a_f(n)\, q^n \in S_k$ **be a normalised Hecke eigenform.**

$$T_m[f] = a_f(m)\, f$$

Fourier coefficients

- $a_f(mn) = a_f(m)a_f(n) \quad (\gcd(m,n) = 1)$
- $a_f(m)a_f(n) = \sum_{d \mid \gcd(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right)$
- $a_f(p)a_f(p^\ell) = a_f(p^{\ell+1}) + p^{k-1} a_f(p^{\ell-1})$

Hecke operators

- $T_m \circ T_n = T_{mn} \quad (\gcd(m,n) = 1)$
- $T_m \circ T_n = \sum_{d \mid \gcd(m,n)} d^{k-1}\, T_{\frac{mn}{d^2}}$
- $T_p \circ T_{p^\ell} = T_{p^{\ell+1}} + p^{k-1}\, T_{p^{\ell-1}}$

Big picture

Theorem (Atkin–Lehner)

If $f(z) = q + \sum_{n=2}^{\infty} a_f(n)q^n \in S_{2k}(\Gamma) \cap \mathbb{Z}[[q]]$ *is a newform, then the following are true:*

1. *If* $\gcd(m, n) = 1$, *then* $a_f(mn) = a_f(m)a_f(n)$.
2. *If* $p \nmid N$ *is prime and* $\ell \geqslant 2$, *then*

$$a_f(p^\ell) = a_f(p)a_f(p^{\ell-1}) - p^{2k-1}a_f(p^{\ell-2}).$$

Higher Level Modular Forms

Congruence Subgroups

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) = \{ \gamma \in \Gamma_0(N) \ : \ a \equiv d \equiv 1 \pmod{N} \}$$

Cusps

Cusps are equivalence classes of points in

$$\mathbb{Q} \cup \{\infty\}$$

under the action of the congruence subgroup.
At higher level, there are multiple cusps, each with its own Fourier expansion.

In the level 1 case, there was only one equivalence class with ∞ as a representative, so we had only one cusp.

Modular Forms

A modular form of weight k and level N for Γ satisfies

$$f(\gamma z) = (cz + d)^k f(z)$$

for every

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

It is holomorphic on $\mathbb{H}$ and at all cusps.

Cusp Forms

A cusp form is a modular form which vanishes at every cusp.

$$S_k(\Gamma) \subseteq M_k(\Gamma)$$

Oldforms and Newforms for $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$

The Issue

- $S_k(\Gamma)$ has a basis of eigenforms for all T_n with $\gcd(n, N) = 1$, but not necessarily for all T_p when $p \mid N$.
- This happens because some forms in $S_k(\Gamma)$ come from *lower levels*, creating larger eigenspaces.

Idea (Atkin–Lehner)

- Oldforms (from lower levels)
- Newforms (genuinely level N)
- $S_k(\Gamma(d)) \subseteq S_k(\Gamma(N))$ if $d \mid N$
- $f \in S_k(\Gamma(M)) \Rightarrow f(dz) \in S_k(\Gamma(N))$ if $d \mid N/M$ and for every proper divisor M of N

Oldforms

$$S_k^{\mathrm{old}}(\Gamma) := \mathrm{span}\left\{ f(dz) : \begin{array}{c} f \in S_k(\Gamma), \\ M \mid N,\ M < N \\ d \mid N/M \end{array} \right\}$$

$$\subseteq S_k(\Gamma)$$

Petersson inner product

$$\langle f, g \rangle = \frac{1}{[SL_2(\mathbb{Z}) : \overline{\Gamma}]} \iint_{\mathcal{F}} \mathrm{Im}(z)^{k-2} f(z) \overline{g(z)}\, dx\, dy$$

Newforms $S_k^{\mathrm{new}}(\Gamma) := \left(S_k^{\mathrm{old}}(\Gamma)\right)^{\perp}$

- $S_k(\Gamma) = S_k^{\mathrm{old}}(\Gamma) \oplus S_k^{\mathrm{new}}(\Gamma)$
- Eigenforms for all T_n, $\gcd(n, N) = 1$
- Can normalize: $a_f(1) = 1$
- Then $T_n[f] = a_f(n) f$

Bibliography

Fred Diamond, Jerry Shurman
A First Course in Modular Forms.
Springer (2010).

Henri Cohen, Fredrik Stromberg
Modular Forms A Classical Approach.
American Mathematical Society (2017).

M. Ram Murty, Michael Dewar, Hester Graves
Problems in the Theory of Modular Forms.
Springer Singapore (2016).

Tom M. Apostol
Modular Functions and Dirichlet Series in Number Theory.
Springer Science Business Media (2012)

A. O. L. Atkin, Joseph Lehner
Hecke Operators and Newforms.
Mathematische Annalen (1970).

A. O. L. Atkin, Joseph Lehner
Hecke Operators on $\Gamma_0(m)$.
Mathematische Annalen 185 (1970), 134–160.

Erich Hecke
Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I.
Mathematische Annalen 114 (1937), 1–28.

Srinivasa Ramanujan
On Certain Arithmetical Functions.
Transactions of the Cambridge Philosophical Society 22 (1916), 159–184.

Thank you! Questions?